Low temperature/high density behavior of a Bose gas

For a boson gas the average number of particles in a bath of chemical potential \( \mu \) and thermal bath \( \beta \) is

\[
\bar{N} = \frac{\Xi \Xi}{\beta} = \frac{\Xi}{\exp(\beta(\xi - \mu)) - 1}
\]

so \( \bar{N} \) is determined by \( \mu \).

For a canonical ensemble (N bosons in a container), the above expression holds due to equivalence of ensembles for large \( N \rightarrow \infty \) systems.

\[
N = \frac{1}{\Xi} \sum_i \exp(\beta(\xi_i - \mu)) - 1
\]

for distribution in continuous

\[
N = \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{\omega(\xi) \, d\xi}{\exp(\beta(\xi - \mu)) - 1}
\]

\( \omega(\xi) \) density of states

where \( \omega(\xi) = \frac{d\phi(\xi)}{d\xi} \).

\[
\phi(\xi) = \frac{4\pi}{3} \left( \frac{\sqrt{2m\xi}}{\hbar} \right)^3 V, \quad \omega(\xi) = \frac{4\pi}{3} \left( \frac{\sqrt{2m\xi}}{\hbar} \right)^2
\]

Therefore,

\[
\omega(\xi) = 2\pi \left( \frac{\sqrt{2m\xi}}{\hbar} \right)^3 V \xi^{3/2}
\]

Thus \( \bar{N} \) gives \( \mu(N,T) \) as per above equation.

As \( T \) is decreased (increasing \( \beta \)), in order to achieve same value of \( \bar{N} \), we need to decrease \( \mu \) i.e. \( \mu \) will become less negative and closer to 0.

Thus (due to the term \( \xi^{3/2} \)) we will arrive at \( \mu = 0 \) at some non-zero temperature \( T_b \). In fact, \( T_b \) is given by

\[
N = \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \left( \frac{\omega(\xi)}{\exp(\beta(\xi - \mu)) - 1} \right) d\xi
\]
\[
N = \int_{\varepsilon = 0}^{\infty} \frac{2\pi (\frac{\sqrt{2m}}{\hbar})^3 \varepsilon^{1/2}}{\exp \left( \frac{\varepsilon}{kT_B} \right) - 1} \, d\varepsilon
\]

Let \( x = \varepsilon / kT_B \)

\[
N = 2\pi \left( \frac{\sqrt{2m}}{\hbar} \right)^3 V \int_{x = 0}^{\infty} \frac{x^{1/2}}{\exp(x) - 1} \, dx \left( kT_B \right)^{3/2}
\]

\[
N = 2\pi \left( \frac{\sqrt{2m}}{\hbar} \right)^3 V \left( kT_B \right)^{3/2} \int_{x = 0}^{\infty} \frac{x^{1/2}}{\exp(x) - 1} \, dx
\]

\[
N = 2\pi \left( \frac{\sqrt{2m}}{\hbar} \right)^3 V \left( kT_B \right)^{3/2} \times 1.306 \pi^{1/2}
\]

\[
\left( \frac{1}{2.612 \pi^{3/2}} \right) \left( \frac{N}{V} \right) \left( \frac{l^2}{2m} \right)^{3/2} = \left( kT_B \right)^{3/2}
\]

\[
kT_B = \frac{\hbar^2}{2m} \left( \frac{N}{V} \right)^{2/3} \left( \frac{1}{2.612 \pi^{1/2}} \right)^{2/3}
\]

Note the analogy to fermion gas:

\[
\mu = kT_F = \frac{\hbar^2}{2m} \left( \frac{N}{V} \right)^{2/3} \left( \frac{3}{8 \pi} \right)^{1/2}
\]

Just like \( T_F \) is characteristic temperature for fermions,

\( T_B \) is characteristic temperature for bosons.
temperature \( T \) becomes 
\[
\frac{kT}{\hbar^2} \alpha \frac{1}{2m} \left( \frac{N}{V} \right)^{3/2}
\]

This coincides with density, (analogous to \( \bar{n}_0 \) increasing with density).

Now when \( T < T_B \) because \( \bar{n}_0 \) cannot become positive, we can no longer satisfy the equation which ensures correct # of particles - some of the bosons are missing! Where are these? Well, these are in the ground state.

Paradox is understood by looking at form of \( N \) in the continuous energy distribution

\[
N = \int_0^{\infty} \frac{\omega(\varepsilon)}{\exp(\varepsilon/T) - 1} \, \mathrm{d}\varepsilon.
\]

This term has no contribution from \( \varepsilon = 0 \) states: \( \omega(0) \equiv 0 \)

but in actuality, a fraction of \( \bar{n}_0 \) bosons are in the ground state as indicated by value of \( \bar{n}_0 \), average occupation # of ground state

\[
\bar{n}_0 = \frac{1}{\exp(-\beta_0) - 1}
\]

which implies that if the bath were to supply an unlimited number of particles then they end up in the ground state. But for fixed \( N \), we have the following situation:

\[
\bar{n}_0 = \bar{n}_0 + \int_0^{\infty} \frac{\omega(\varepsilon)}{\exp(\varepsilon/T) - 1} \, \mathrm{d}\varepsilon
\]

\[
\bar{n}_0 \approx N \left( 1 - \left( \frac{T}{T_B} \right)^{3/2} \right)
\]

At \( T > T_B \) \( \bar{n}_0 \) is negligible w.r.t \( N \), higher states have higher density of states.
At $T = T_c$ suddenly $\bar{n}_g$ begins to be preferred. As $T$ decreases below $T_c$, more and more particles pile up in ground state until at $T = 0$ all particles are in ground state. 

\[ N = \bar{n}_g + 0 = \bar{n}_g \]

This is known as Bose-Einstein condensation.

Instead of $T$ temperature, similar transition can be reached by increasing the density of particles. There is a critical density above which bosons condense together.

\[ \frac{\overline{n}_g}{N} = 1 - \frac{PB}{P} \quad P > P_B \]

\[ \frac{\overline{n}_g}{N} = 0 \quad P < P_B \]

This is a critical density below which ground state is not preferred, but above which the particles tend to populate their ground state.

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Excitation of one particle from ground state to excited state increases entropy by $k\ln N$ where $N$ is # of distinguishable particles. But when particles are indistinguishable, there unity energy to do this and so for large $N$, entropy change is zero. No entropy gained in the system & so system prefers to minimize energy (such that $A$ is minimized) therefore resulting in all particles occupying ground state.

Thus the condensation of bosons (or their "attraction") results from indistinguishability.

(Particle occupy lower states than they would classically) (due to "attraction" in bosons) (Fermions occupy higher state than they would classically due to Pauli exclusion or repulsion)
Bose-Einstein condensation ⇒ The N-particle gas can be described by one wave-function ⇒ a macroscopic quantum object.

Note: at OK, De Broglie wavelength is infinite. Thus any density of particles would form a condensate.

Unlike the case of dechons in a metal, getting atoms to correlate quantum mechanically is more challenging. Atoms are heavier hence their wavelengths are shorter and hence the density required for condensation needs to be higher by just orders. Note before we reach densities required for such condensation usually atoms/molecules begin to interact (no longer ideal) and may condense into a solid. 

Before Bose-Einstein condensation is achieved, liquid helium is a counter-example where the light mass 4 large nuclei point energy prevent solidification down to zero kelvin.

BECs have been achieved in liquid helium, excitons in solids, and atomic gases.

- Laser cooling
- Superconductivity
- Superfluidity
- BEC of He atoms ⇒ zero viscosity liquid, no dissipation of energy, all atoms are in ground state. However, here, interaction forces are also conserved with BEC. (Not an ideal Bose gas condensate)
- BEC of composite bosons (Cooper pairs of electrons formed by interaction with a phonon)
- No resistance (dissipation) to flow of this condensate